Star-Struck by Fixed Embeddings
Modern Crossing Number Heuristics

GD ’21

Joint work with Markus Chimani and Tilo Wiedera

Max Ilsen, University of Osnabrück

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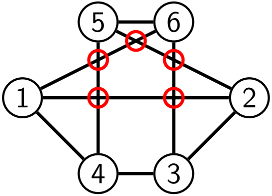
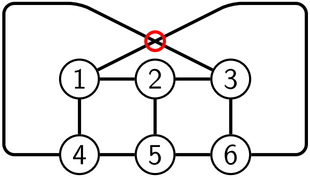
September 15, 2021

Crossing Number Problem

Given: Graph G

Task: Find minimum number of edge crossings in any drawing of G





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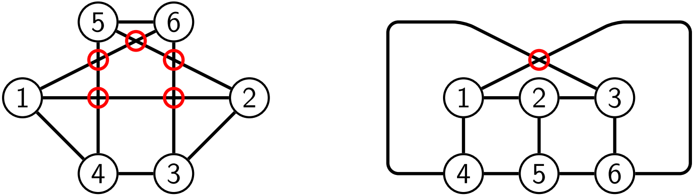


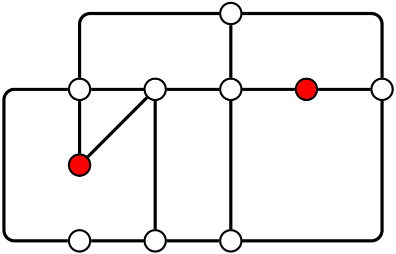
NP-hard even in restricted settings

→ Let's evaluate some heuristics!

Planarization Method

Heuristic #1

→ compute planar subgraph, insert remaining edges iteratively



Batini, Talamo, and Tamassia 1984

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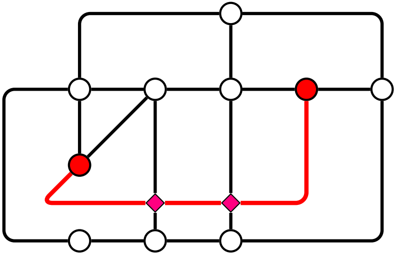

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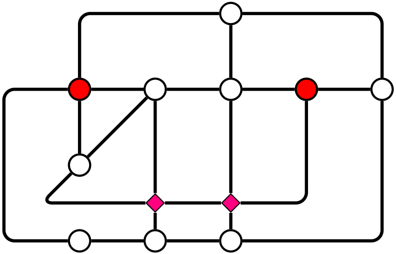

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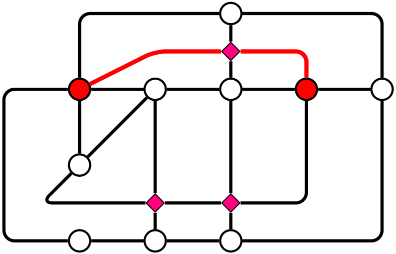

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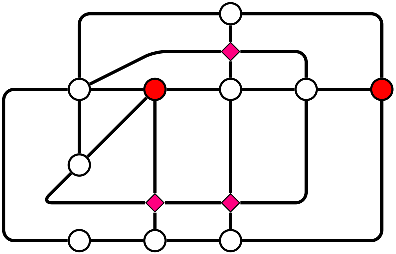

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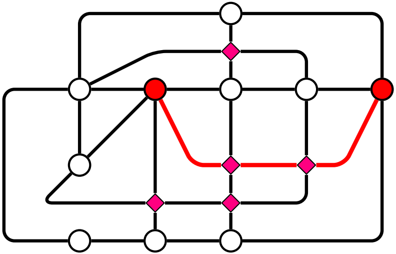

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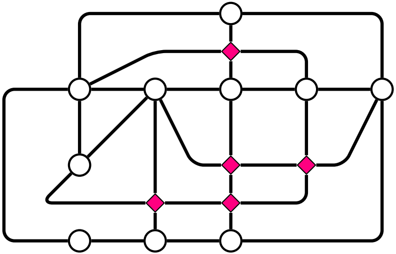

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Planarization Method
Variants

Edge Insertion:

- **fix**ed embedding (Batini, Talamo, and Tamassia 1984)
- **var**iable embedding (Gutwenger, Mutzel, and Weiskircher 2005)
- **multi**ple edge insertion (Chimani and Hlinený 2011)

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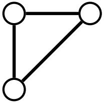
Post-Processing:

- **all**: remove and reinsert every edge at the end (Gutwenger and Mutzel 2003)
- **inc**: perform **all** after each edge insertion (Chimani and Gutwenger 2012)

Chordless Cycle Method

Heuristic #2

→ compute chordless cycle, insert partial stars connected to it



[1] Clancy, Haythorpe, and Newcombe 2019

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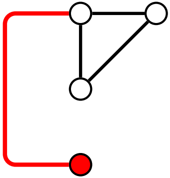

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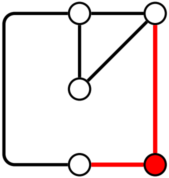

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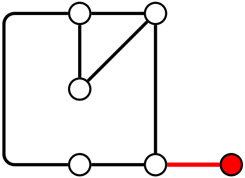

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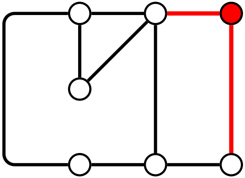

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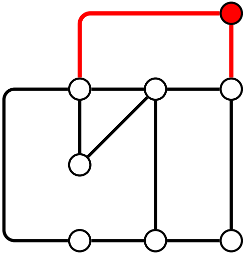

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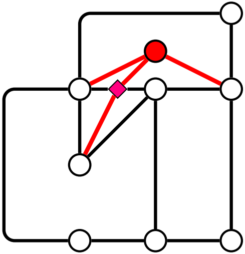

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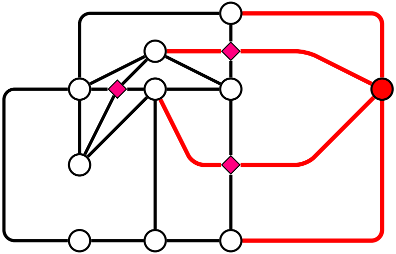

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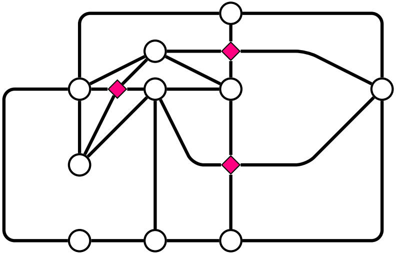

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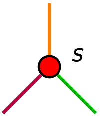
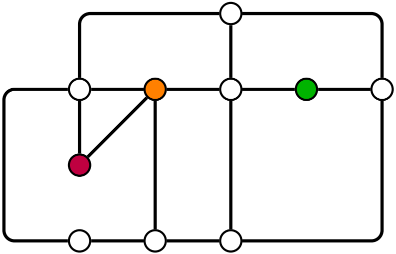
Given: Planar graph G, embedding Π of G, star s not yet in G

Task: Insert s into Π s.t. the number of crossings in Π is minimized





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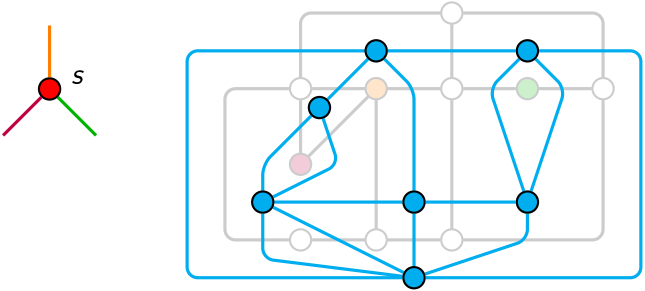

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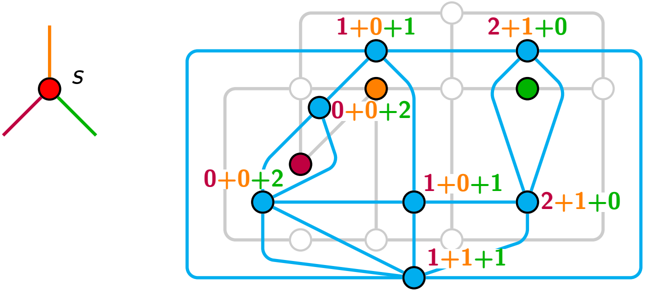

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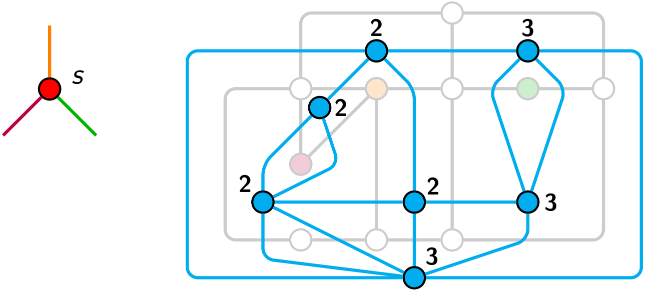

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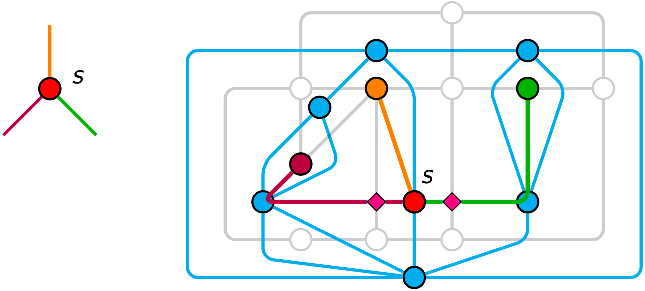

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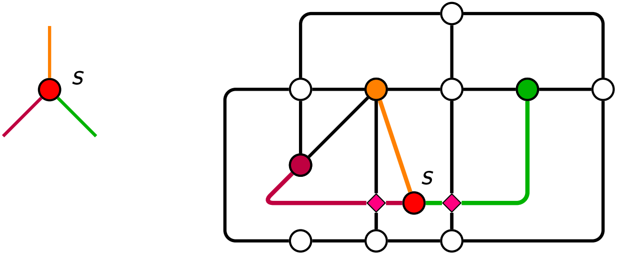

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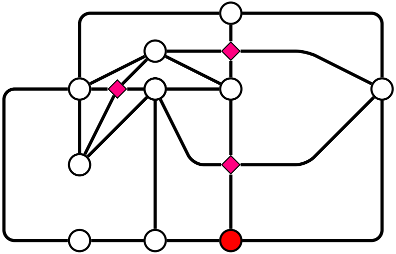
Star Reinsertion Method

Post-Processing

→ create initial planarization, remove and reinsert stars until no reinsertion can decrease number of crossings



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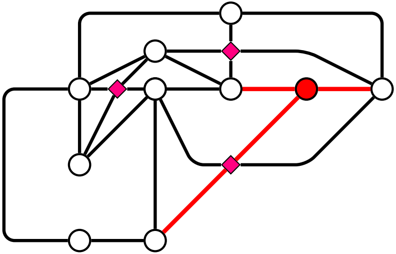

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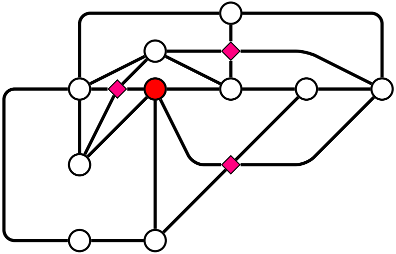

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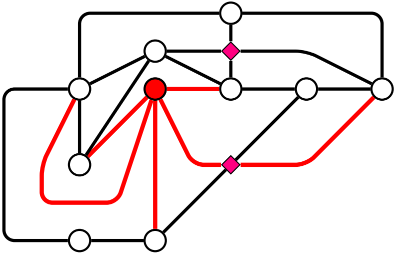

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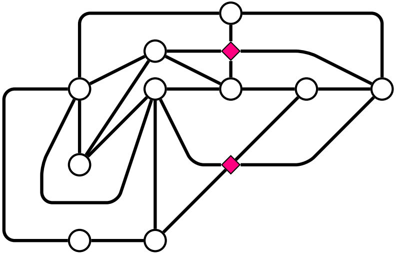

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Heuristic #3: Our Own Approach

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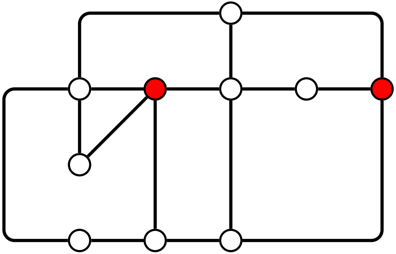

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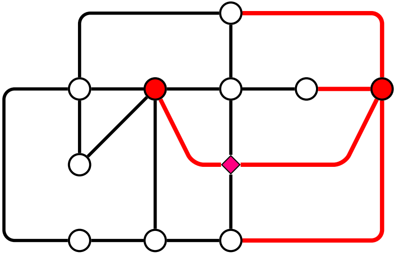

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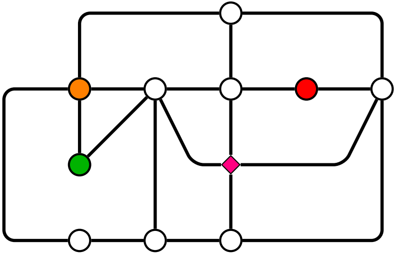

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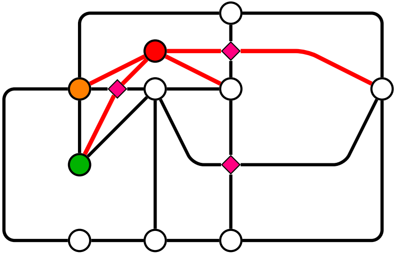

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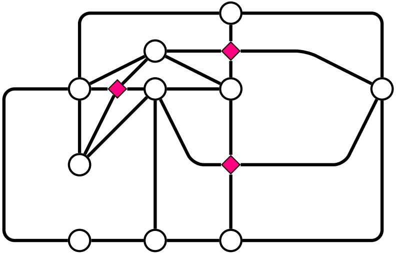

Evaluation

Setup:

- C++ (GCC 8.3.0), Open Graph Drawing Framework (OGDF)
- single physical processor of a Xeon Gold 6134 CPU (3.2 GHz)
- memory limit of 4 GB, no time limit

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Pre-Processing:

- non-planar core reduction[1]
- precomputed planar subgraph and chordless cycle

[1] Chimani and Gutwenger 2009

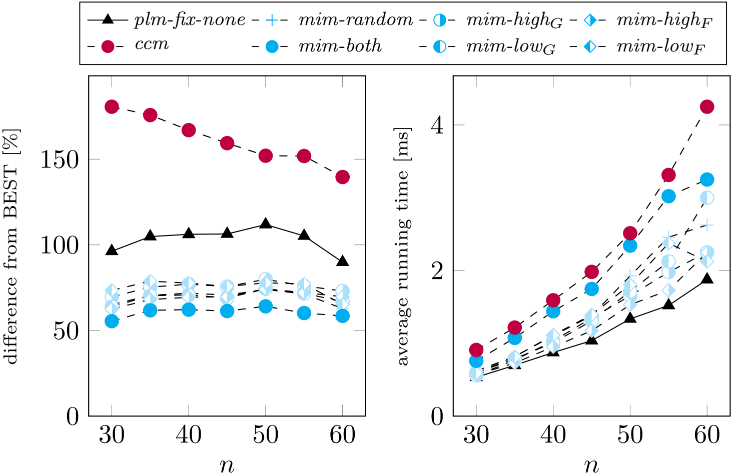
Instances

	$\geq$ 25 vertices in NPC	
Name	#	$\lvert V(G) \rvert$
Rome	3668	25–58
North	106	25–64
Webcompute	75	25–112
Expanders	240	30–100
Circuit-Based	45	26–3045
ISCAS-85	9	180–3045
ISCAS-89	24	60–584
ITC-99	12	26–980

	all instances	
Name	#	$\lvert V(G) \rvert$
Complete	46	5–50
Complete-Bip.	666	10–80
KnownCR	1946	9–250
$C \,\square\, C$	251	9–250
$G \,\square\, P$	893	15–245
$G \,\square\, C$	624	15–250
$P(\_, \_)$	178	10–250

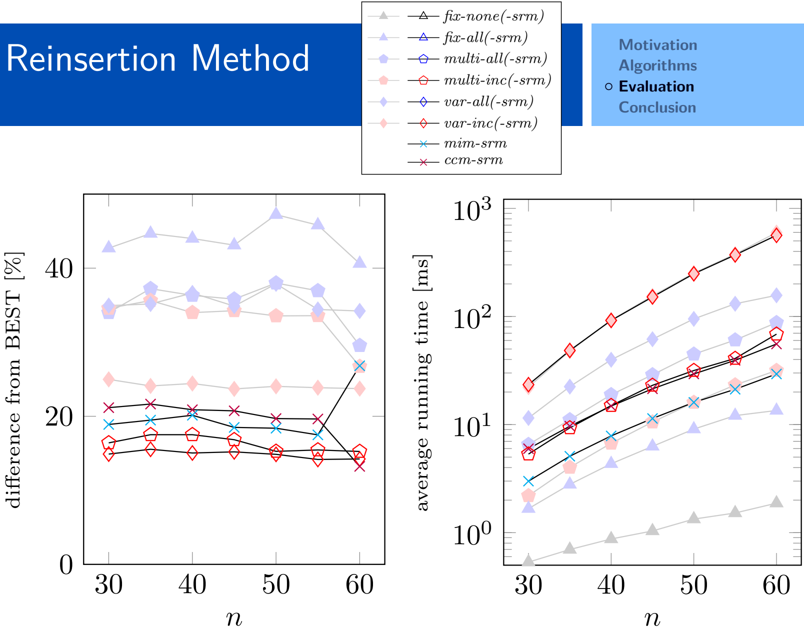
Fast Heuristics

Rome



Star Reinsertion Method

Rome



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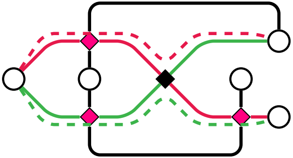
Thank you! Any questions?

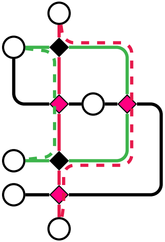
Non-simple Crossings

Removal



α-**crossing**

adjacent edges

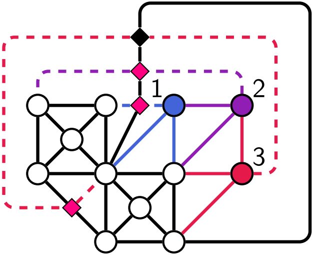


β-**crossing**

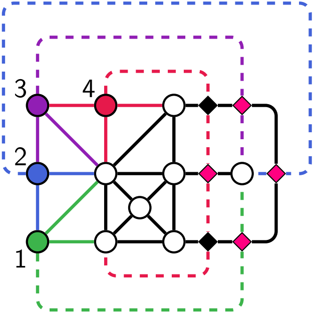
two edges, multiple crossings

Non-simple Crossings

Creation



α-**crossing**



β-**crossing**

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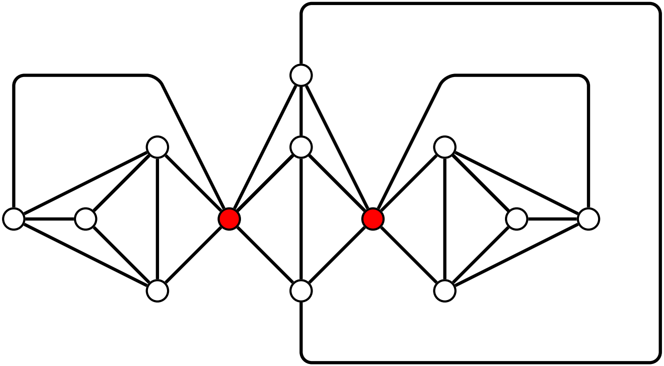

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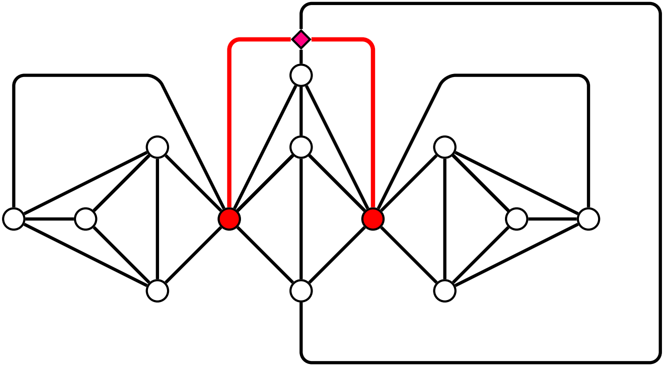

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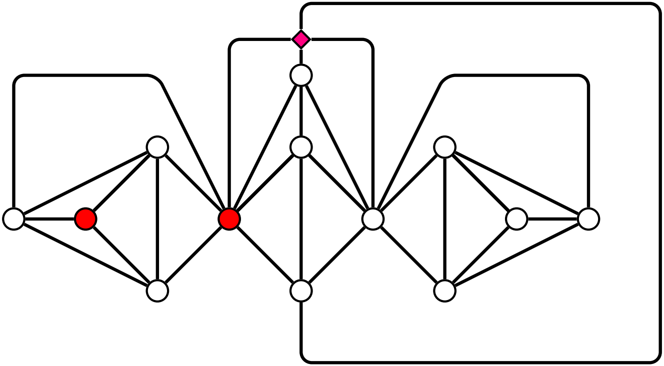

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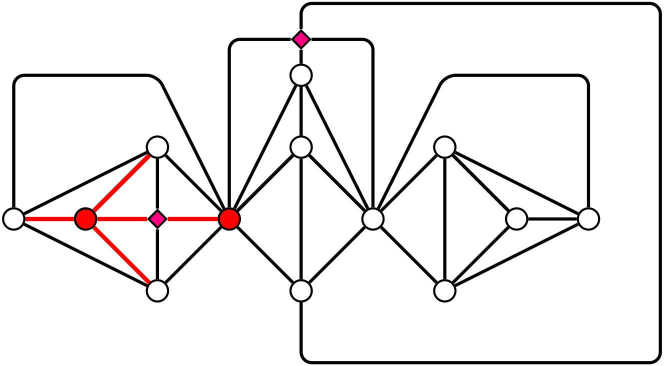

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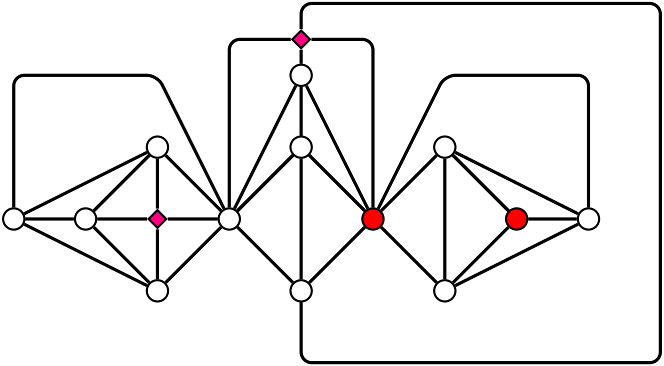

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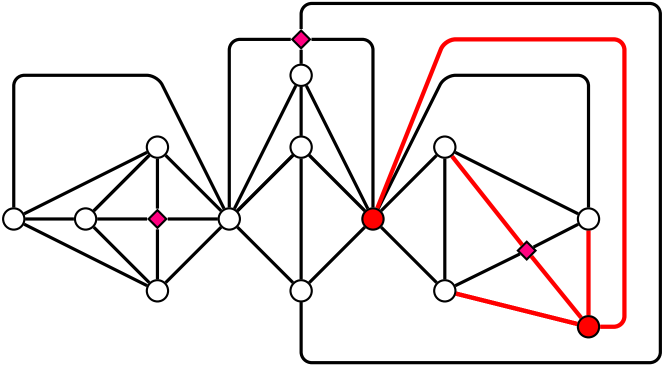

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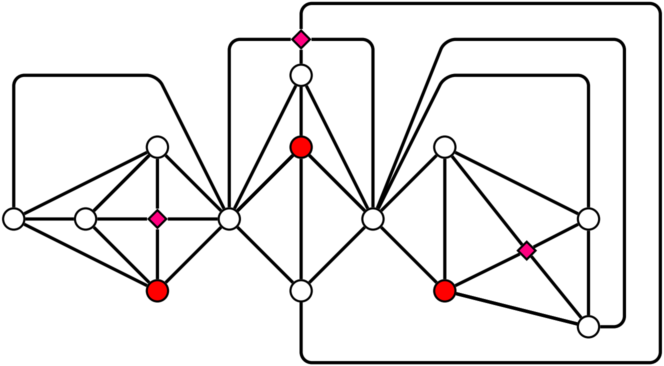

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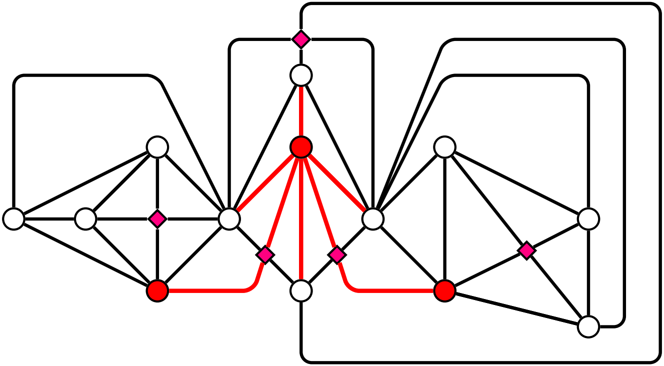
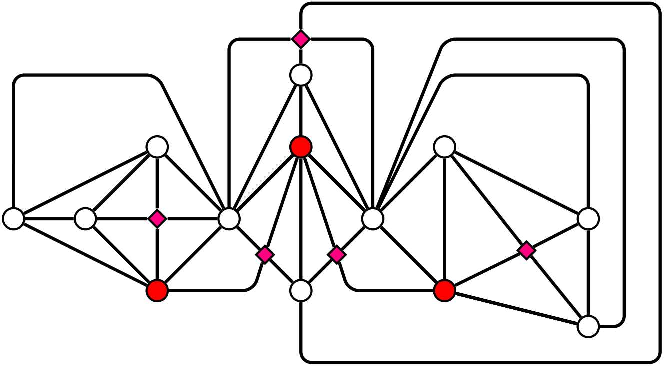


Mixed Insertion Method

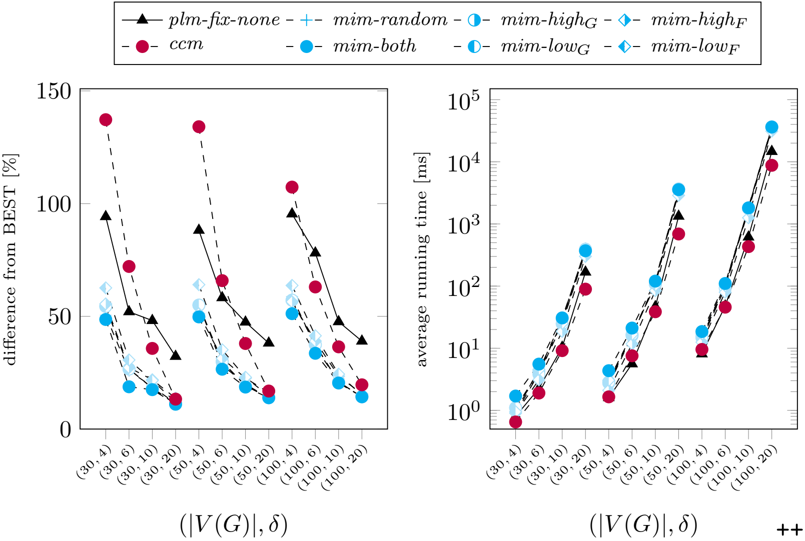
Heuristic #3: Our Own Approach

→ compute planar subgraph, insert remaining edges via star insertion of their endpoints (if possible)



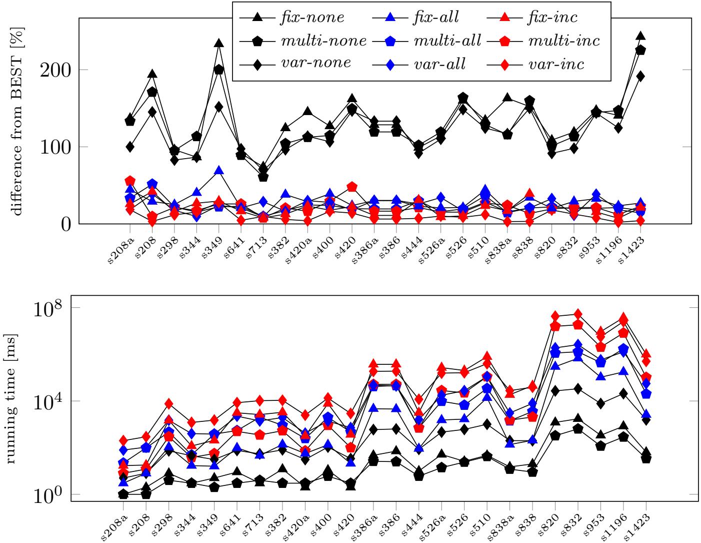
Fast Heuristics

Expanders



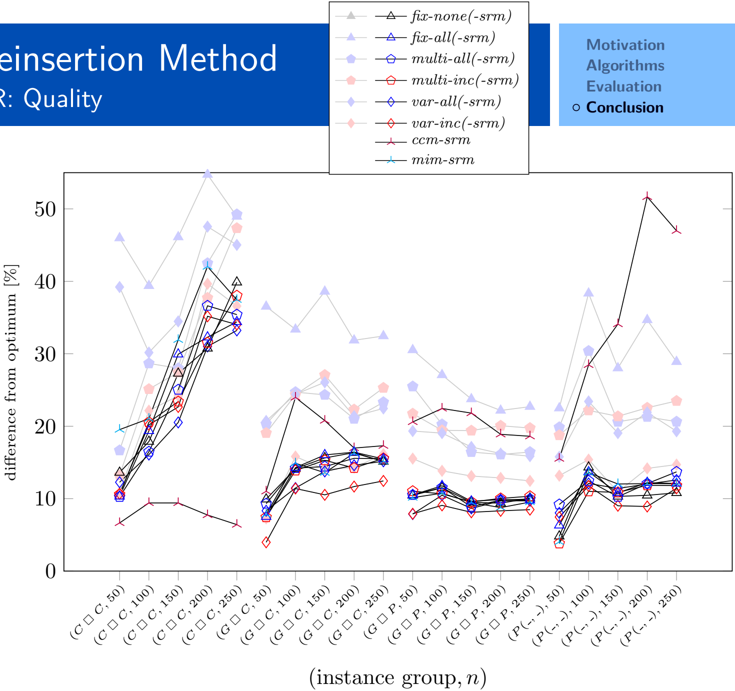
Planarization Method

ISCAS-89



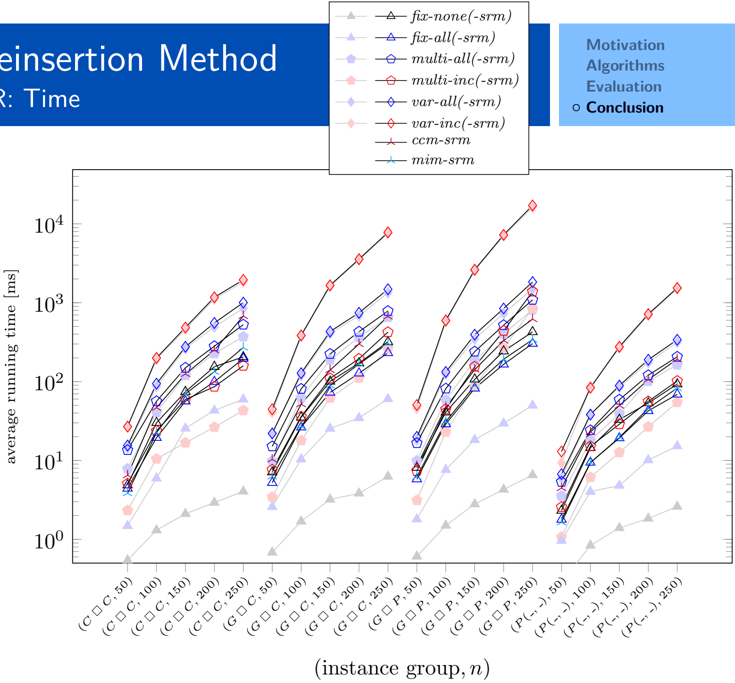
Star Reinsertion Method

KnownCR: Quality



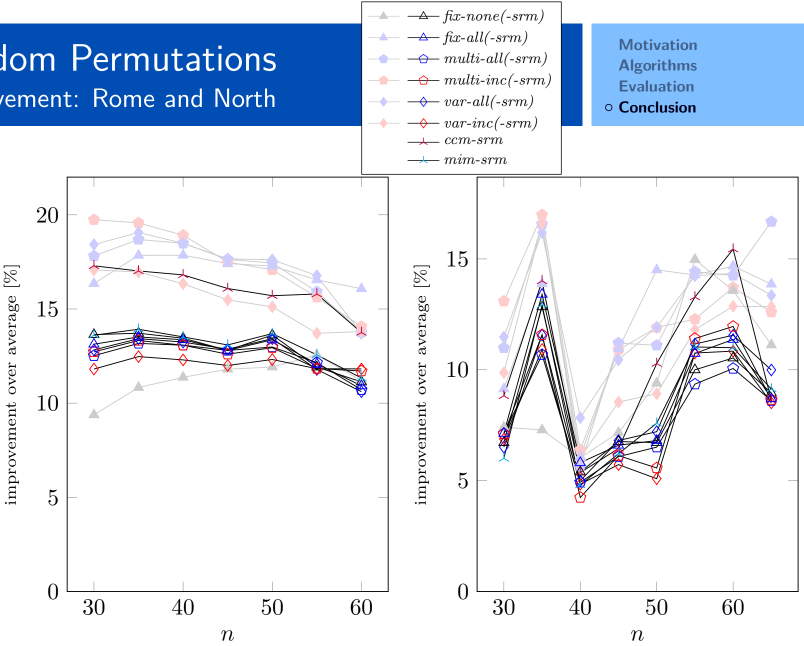
Star Reinsertion Method

KnownCR: Time



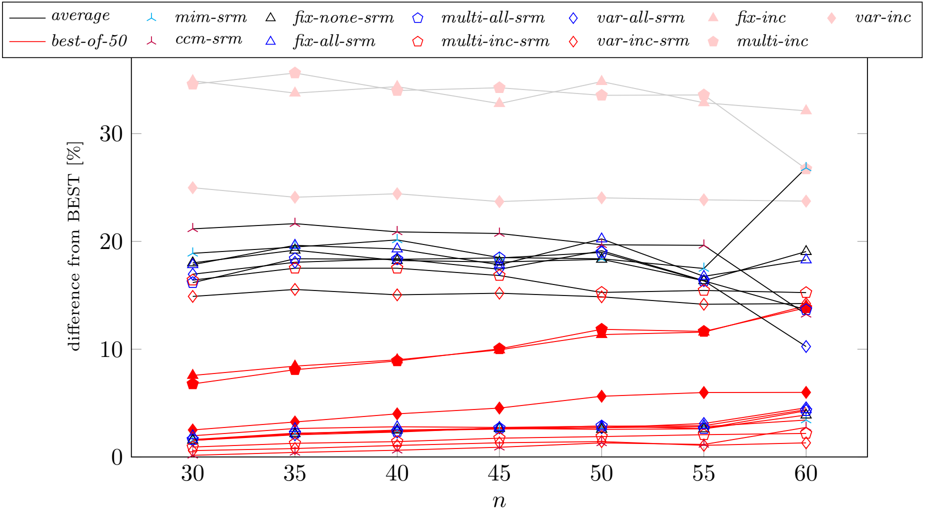
Random Permutations

Improvement: Rome and North



Random Permutations

Rome



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